Mobility edge scaling at semiclassical and dissipative Hall transitions

Joel E. Moore
Department of Physics, University of California, Berkeley, CA 94720
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The universal anomalous diffusion scaling is obtained for the semiclassical quantum Hall transition, which has been argued to describe samples with dissipation or correlated impurities. The results explain a discrepancy between existing numerics and the expected scaling $\eta = 2x_1 = \frac{1}{2}$, which is violated because of a cancellation in the scaling function. The crossover with increasing observation time from semiclassical to quantum scaling is shown to explain a recent experiment which finds different scaling laws depending on how the localization length is extracted.

The electronic eigenstates of disordered systems at critical points such as the quantum Hall plateau transition[1] or metal-insulator transition[2] are believed to have fractal structure described by universal scaling laws, but analytic results for such scaling laws are quite scarce. This paper finds the exact anomalous diffusion scaling in a standard semiclassical model for the quantum Hall transition, extending previous numerical studies[3]. Several authors[4,5,6] have argued that this semiclassical limit is relevant to various experimental situations, and it also shares some features with the spin quantum Hall transition[7,8] in disordered superconductors. Here the semiclassical model is compared to recent experiments[9] which appear to show both "quantum" and "classical" localization scaling laws in the same sample, depending on how the localization length is extracted.

The semiclassical model has recently appeared in studies of how dissipative effects can modify the quantum Hall transition[4,5]. Many experiments observe localization length scaling more consistent with the classical value $\nu = \frac{4}{3}$ than the expected quantum value $\nu \approx 2.35 \pm 0.05$[10,11]. By suppressing tunneling and introducing a finite dephasing length, dissipation increases the range of chemical potentials where a semiclassical description applies, but causes simple diffusion over a nonzero range around the critical chemical potential. The main experimental results can be understood in a simple picture incorporating dissipation, without requiring a new dissipation-dominated critical point other than diffusion.

The starting point of the analysis is a standard lattice model for classical motion on percolation hulls or random level surfaces; quantum mechanics only enters this model in properties like the density of states and conductivity[3,12] which count the number of trajectories. The random level surface problem was first introduced in the quantum Hall context as a useful but quantitatively incorrect model for the integer transition[13,14]. Our approach uses a mapping to a class of lattice polymers[15] to find time-dependent correlations in the semiclassical random level surface problem and resolve a disagreement between existing numerics and analytical work.

The results improve upon Monte Carlo calculations of Evers[3] on the random level surface problem, which correctly found a deviation from simple scaling but did not reach the extremely long paths ($\geq 10^4$ disorder correlation lengths) where the true asymptotic scaling sets in. There is a universal anomalous diffusion exponent $\eta = \frac{1}{4}$ which characterizes the non-Gaussian correlation of critical states. This result differs from the value $\eta = \frac{1}{2}$ predicted by a simple scaling argument because the leading scaling term vanishes. The restricted "open-walk" version of the problem, which is more convenient for numerics[3], is shown analytically to have $\eta = \frac{1}{4}$ with divergent logarithmic corrections resulting from a short-distance singularity in the associated polymer ensemble. Monte Carlo simulations are used to verify some of the predicted scaling behavior. Aside from their direct relevance to quantum Hall transitions, the results suggest possible features of other disordered electronic transitions: cancellations in scaling functions, logarithmic corrections, and slow decay of finite-size effects.

The existence of anomalous scaling (nonzero η) for the ordinary integer transition was shown by Chalker and Daniell[1]: in the scaling regime $q, \omega \ll 1$, the spectral function has two distinct universal limits:

$$S(\mathbf{q}; E_c, \omega) \sim \begin{cases} q^2\omega^{-2} & \text{if } q^2 \ll \omega \\ \omega^{-\eta/2} q^{\eta-2} & \text{if } q^2 \gg \omega \end{cases}. \qquad (1)$$

This form satisfies the scaling law $S(q; E_c, \omega) = \omega^{-1} f(q^2/\omega)$ which follows from a homogeneity assumption. $S(q; E_c, \omega)$ is defined as the Fourier transform of

$$S(\mathbf{r}; E_c, \omega) = \Big\langle \sum_{i,j} \delta(E_i + \frac{\omega}{2} - E_c)\delta(E_j - \frac{\omega}{2} - E_c) \times \psi_i(0)\psi_i^*(\mathbf{r})\psi_j(\mathbf{r})\psi_j^*(0) \Big\rangle. \qquad (2)$$

We now define the density-density correlation $\langle \rho(0,0)\rho(r,t) \rangle$ at $E = E_c$ in the random level surface model, with scaling analogous to (2).

Consider a classical charged particle moving in the **x**-**y** plane in a magnetic field $B\hat{\mathbf{z}}$ and random potential $V(x)$. For a smoothly varying potential, the particle velocity averaged over the fast cyclotron motion is $\mathbf{v} = (\mathbf{E} \times \mathbf{B})/B^2$. The particle velocity is perpendicular to the potential gradient, and the particle moves along constant energy surfaces of the random potential. For a uniformly distributed potential $V(x) \in [-1, 1]$, the typical size of level surfaces at energy E diverges as $E \to 0$. The connection to percolation comes about because a level surface at

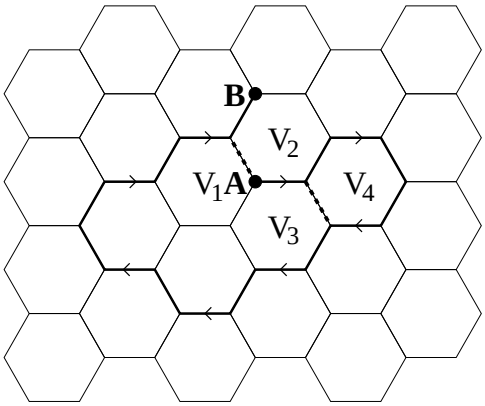

FIG. 1: The dotted edges are self-contacts: the edge between V_3 and V_4 is an antiparallel self-contact, while that between V_1 and V_2 is a parallel self-contact. This path is not allowed classically since the walk passes V_2 both on the right and on the left.

energy E separates regions with $V > E$ from those with $V < E$. The level surfaces are closed non-self-intersecting loops, whose statistical properties are exactly the same as percolation hulls[13], or self-interacting ring polymers at the critical θ-point[16].

The results from percolation and polymers used to study this problem require discretizing the motion so that the particle moves on a regular lattice. From numerics[3,12] it is known that the particle has nonzero mean velocity at the critical energy, so time can be discretized as well: the particle takes one step on the lattice per time unit. The resulting model on the hexagonal lattice[12,15] is depicted in Fig. 1. There is an independent random potential V_i on each face of the lattice. A particle of energy E moves so that the potential of faces to its left (right) is always greater (less) than E.

Averaging over disorder, the probability to be at b after N steps starting from a can be written as a weighted sum over self-avoiding walks (SAWs) or loops (SAPs)[15]. The result is (H is the number of different hexagons visited by the SAW or SAP)

$$f(r,N) \;\propto \sum_{\substack{\text{SAPs } i \text{ through } a \text{ and } b,\\ l=\text{ length of SAP},\\ q=\text{ steps from } a \text{ to } b}} \delta_{N \bmod l,q} 2^{-H} + \sum_{\substack{\text{SAWs } j \text{ of length } N\\ \text{from } a \text{ to } b\\ \text{no } \| \text{ self}-\text{contacts}}} 2^{-H}. \qquad (3)$$

Here each SAP should be summed twice, once with distance q and once with distance $l-q$. The classical analogue of the spectral function $S(q,\omega)$ is the imaginary part of (passing to continuous time)

$$\Pi(q,\omega) \equiv -i \int d\mathbf{r} \int_0^\infty dt\, e^{-i(\mathbf{q}\cdot\mathbf{r}+\omega t)} f(r,t). \qquad (4)$$

For random walks $\Pi(q,\omega) \propto (\omega - iDq^2)^{-1}$ and $\omega\Pi$ is a function of q^2/ω (diffusive scaling); such scaling is violated at the semiclassical Hall transition, as seen below.

The weight 2^{-H} in (3) is exactly that of the θ' model studied by Duplantier and Saleur[16], which is in the same universality class as the ordinary θ-point. The critical exponents γ and ν_θ, defined through

$$\sum_{\text{SAWs of length N}} 2^{-H} \;\sim\; \mu^N N^{\gamma-1}$$
$$\frac{\sum_{\text{SAWs of length N}} R^2 2^{-H}}{\sum_{\text{SAWs of length N}} 2^{-H}} \;=\; \langle R^2 \rangle_{\text{SAW}} \sim N^{2\nu_\theta}, \qquad (5)$$

take values $\gamma = \frac{6}{7}$, $\nu_\theta = \frac{4}{7}$ for this class, and $\mu = 1$ on the hexagonal lattice[16]. Here and in the following $\langle\rangle_{\text{SAW}}$ denotes an average over self-avoiding walks, while $\langle\rangle$ denotes an average over disorder configurations in the random level surface problem.

The linear increase with time of the mean squared particle displacement at E_c follows from the values of polymer exponents ν_θ and γ: $\langle R^2(N)\rangle \propto N^{\gamma-1+2\nu_\theta} = N$. Higher moments of the particle distribution function show nontrivial scaling laws:

$$\langle R^{2n}(N)\rangle \propto N^{\gamma-1} N^{2n\nu_\theta} = N^{(8n-1)/7}, \quad n \geq 1. \qquad (6)$$

For random walks $\langle R^{2n}(N)\rangle \propto N^n$. The higher moment laws (6) reflect the non-intersection and memory properties of random level surfaces, which lead to a non-Gaussian distribution of $f(r,N)$.

The expression (3) for the particle distribution function after N steps includes both SAWs and SAPs. Since the sum over SAPs contains not just loops of length N but of all shorter lengths, this term contributes a large constant background which is difficult to subtract numerically. The most comprehensive numerics have been performed on a reduced problem including only the average over self-avoiding walks. It follows from simple properties of a polymer scaling function that Monte Carlo numerics suggesting $\eta \approx 0$ for the reduced problem are not reaching the asymptotic regime: the actual value is $\eta = \frac{1}{4}$ with a logarithmic prefactor. Then a similar argument gives $\eta = \frac{1}{4}$ for the full problem (open and closed polymers), with no logarithmic corrections.

The basic scaling law for polymers predicts that the particle distribution function f_0 for the reduced problem has the form

$$f_0(\mathbf{r},t) = t^{-d\nu_\theta} H(rt^{-\nu_\theta}) \qquad (7)$$

where H is normalized as $\int H(x)\, d^d x = 1$. H is positive, falls off rapidly as $x \to \infty$, and is smooth except for the origin, where $H(x) \sim x^{(\gamma-1)/\nu_\theta} = x^{-1/4}$. This behavior at small x is the simplest example of a polymer contact exponent: a new critical exponent describes the scaling as the two ends of the polymer approach each other. The value $(\gamma-1)/\nu_\theta$ can be derived from noting that as the ends approach each other, a SAW becomes a SAP. This

divergence at small x is the source of the logarithmic corrections in the reduced problem.

It is convenient to isolate the divergence in $H(x)$:

$$H(x) = cx^{-1/4}e^{-x^2/2} + H_{\rm reg}(x), \tag{8}$$

where c is a positive constant and $H_{\rm reg}$ is some smooth function (no longer necessarily positive). The cutoff on the singular part is arbitrary; a Gaussian is chosen for simplicity. The distribution function in momentum space is

$$\begin{aligned}\hat{f}_0\ (\mathbf{q}\ ,t) &= \int e^{-i\mathbf{q}\cdot\mathbf{r}} f_0(\mathbf{r},t)\, d^2\mathbf{r} \\ &= 2\pi\int_0^\infty J_0(qr)H(rt^{-4/7})t^{-8/7} r\, dr \equiv \hat{f}_0(qt^{4/7}) \end{aligned} \tag{9}$$

The contribution of the regular part $H_{\rm reg}$ to $f(q,\omega)$ is finite (possibly zero) as $\omega \to 0$:

$$\begin{aligned} f_{\rm reg}(q,0) &\equiv 2\pi\int_0^\infty\int_0^\infty J_0(uqt^{4/7})H_{\rm reg}(u) u\, du\, dt \\ &= \frac{7\Gamma(7/8)^2\sqrt{\sqrt{2}-1}}{2q^{7/4}}\left(\int_0^\infty \frac{H_{\rm reg}(u)}{u^{3/4}}\, du\right) \end{aligned} \tag{10}$$

The singular part of $H(x)$ gives a logarithmically divergent contribution C as $\omega \to 0$:

$$\begin{aligned} C &= 2\pi c\int_0^\infty\int_0^\infty e^{-i\omega t} J_0(uqt^{4/7}) u^{-1/4} e^{-u^2/2} u\, du\, dt \\ &= 2\pi c\Gamma(7/8)2^{-1/8}\int_0^\infty e^{-i\omega t} F(7/8,1,-q^2t^{8/7}/2)\, dt \\ &= \frac{2c\Gamma(7/8)^2\sqrt{\sqrt{2}-1}}{q^{7/4}}\left(\log(q^{7/4}/\omega) + \ldots\right) \end{aligned} \tag{11}$$

where the omitted terms are finite as $\omega \to 0$ and F is the confluent hypergeometric function.

From (11) Im Π_0 scales as $q^{-7/4} = q^{-2+\eta}$ with $\eta = \frac{1}{4}$, with a logarithmically divergent prefactor resulting from the polymer contact singularity $H(x) \sim x^{-1/4}$. The singularity is easily missed numerically since it only starts to dominate the scaling function for small values of x in dimensionless units, so extremely long walks with $N \geq 10^4$ are required for its observation. We have performed Monte Carlo simulations of walks up to $N = 10^5$ to verify the predicted increase of $H(x)$ at small x.

Now consider the average over all (closed and open) paths which gives Im Π. After a long time t, only a fraction $t^{-\frac{1}{7}}$ of paths are open, and most particles move on closed loops. Naïve scaling predicts that $\eta = 2x_1 = \frac{1}{2}$, but the scaling function is shown below to vanish in the limit $\omega \to 0$. The next term is nonvanishing and gives Im $\Pi \sim \omega^{1/7}q^{-7/4}$, so $\eta = 1/4$. There are no logarithmic corrections, unlike in the open-walk case, because of the $t^{-\frac{1}{7}}$ damping of open walks.

As $t \to \infty$ the particle distribution function goes to a nonzero limit

$$f(r,\infty) \equiv \sum_{\substack{\text{closed paths} \\ \text{including 0 and } r}} 2^{-H} L^{-1}, \tag{12}$$

	ν	σ	η
CQHT	$\frac{4}{3}$[13]	$\frac{\sqrt{3}}{4} \approx 0.433$[18]	$\frac{1}{4}$
IQHT	2.35 ± 0.05[11]	0.5 ± 0.1[19]	0.38 ± 0.04[1]
SQHT	$\frac{4}{3}$[8]	$\frac{\sqrt{3}}{4}$[18]	?

TABLE I: Comparison of exact critical properties of semiclassical Hall transition with the ordinary integer transition (IQHT) and spin transition (SQHT). Critical conductivity is in units of $\frac{e^2}{h}$ per spin.

with $\sum_r f(r,\infty) = 1$ and L is the path length. The background contributes a $\delta(\omega)$ part which is henceforth ignored (difficulty in separating $\delta(\omega)$ is the reason why Monte Carlo calculations are often performed on the reduced open-walk problem). Defining $\tilde{f}(r,N) \equiv f(r,N) - f(r,\infty)$, some straightforward cancellations show that $\sum_N \tilde{f}(r,N) = 0$ for all r, so for all q, in the continuum limit $\int_0^\infty \tilde{f}(q,t)\, dt = 0$ and Im $\Pi \to 0$ as $\omega \to 0$.

For nonzero $\omega \ll q^{7/4}$ the contribution of paths shorter than ω^{-1} is nearly zero from the same cancellations; long paths determine

$$\text{Im } \Pi \sim \int_{\omega^{-1}}^\infty t^{-1/7}\cos(\omega t - 1) g(qt^{4/7})\, dt \sim \frac{\omega^{1/7}}{q^{7/4}}. \tag{13}$$

The upshot of the above results is that for neither the open-walk case nor the full problem is the scaling trivial: logarithmic corrections appear in the open-walk case, and the scaling function in the full problem vanishes for small argument, leading to the emergence of a new power-law and $\eta = \frac{1}{4}$ instead of $\eta = 2x_1 = \frac{1}{2}$. A similar discrepancy between η and the value of $2x_1$ obtained from finite-size scaling for the integer Hall transition was noted in the original paper[1].

An interesting question is whether the spin quantum Hall transition (SQHT), which has the same σ and ν as the semiclassical transition (Table I) assuming a simple relation between Landauer and Einstein conductances[12], also has the same η. Some sums over paths in the SQHT network model are exactly equal to sums over percolation hulls[8,17], but individual paths are not directly related to individual hulls, so it is not clear that η need be the same.

Several authors[4,5] have proposed that the semiclassical limit discussed above may be relevant to the many experimental samples which fail to show simple $\nu \approx 2.3$ scaling down to low temperatures. The exponent ν is traditionally obtained by measuring two different combinations of ν and z, where the dynamical exponent z is typically equal to 1 within experimental error. The scaling of the width of the transition region with temperature measures the product νz: experimental results on such scaling show variously scaling down to low temperature with $\nu z \approx 2.3$[10], scaling down to low temperature with $\nu z \approx 1.5$[9], and a breakdown of scaling at low temperature with high-temperature scaling $\nu z \approx 1.5$[20].

An alternate way to determine the localization length and hence ν directly, without measuring z, is via variable-

range hopping in the localized regime far from the transition. Recent experiments[9] found for three samples one scaling law for the plateau width, corresponding to $\nu \approx 1.5$ if $z = 1$, and another corresponding to $\nu \approx 2.3$ for the localization length obtained from variable-range hopping at finite temperature. Other samples had plateau width ν values ranging from 1.3 to 2.2. This surprising appearance of different power laws in the same samples suggests either a nontrivial value for $z \neq 1$ or that the two measurements are probing different physics. The remainder of this paper discusses an interpretation of these experiments based on quantum-mechanical tunneling between semiclassical states and a loss of phase coherence near the critical energy. Note that the single-particle motion on percolation hulls discussed below is distinct from percolation of macroscopic quantum Hall "puddles"[4].

A possible explanation for the existence of two scaling laws in the same samples is that the plateau width measurements, taken at high current so that each electron state is only briefly occupied, do not see the effects of the relatively slow quantum tunneling processes, which cause $\nu \approx 2.3$ for the true zero-temperature, infinite-time localization length. The effective localization length on short time scales should then be described by the classical percolation exponent $\nu = \frac{4}{3}$. In[9] the plateau width measurements are taken at conductivity σ of order $10^{-4}e^2/h$, while the variable-range-hopping localization length is taken from data over the range $10^{-13} \leq \sigma h/e^2 \leq 10^{-5}$. This argument predicts is that the localization length extracted from plateau width scaling should be shorter than that from variable-range hopping at low current.

An area of current interest is how dissipation via coupling to low-energy excitations, such as weakly localized electrons, can modify quantum phase transitions[5]. Such low-energy excitations of unknown origin seem experimentally to generate a finite dephasing length down to the lowest measured temperatures in some samples. As the chemical potential nears the critical energy, power-law scaling of the localization length requires phase coherence on increasing length and time scales. The physics of the quantum Hall fixed point depends on both tunneling and phase coherence, which keeps the states from being truly extended except at the critical energy. Once the localization length is larger than the dephasing length, the transport should be diffusive (finite σ_{xx}), as seen in some samples in a finite range around the critical energy.

Another explanation for classical percolation scaling is that a smooth disorder potential (disorder correlation length larger than the magnetic length) shows a larger crossover region where classical percolation applies than a sharp potential, because tunneling is reduced. However, this would not explain the breakdown in scaling of the plateau width at low temperature observed in many samples, and would require samples from similar growth runs to have very different impurity distributions. A possible direct test of whether changes in dephasing length are indeed correlated with the sample-dependent behavior would be the addition of a controllable coupling to dissipation, as done for low B in[21].

Our main conclusion is that the semiclassical limit of the quantum Hall transition has nontrivial mobility edge scaling which can be found exactly using results from the theory of polymers or percolation. This limit can be experimentally observed in samples with dissipation or smooth potential fluctuations and is closely related to the spin quantum Hall transition.

Note added: After this paper was submitted, a preprint[22] appeared which examines the anomalous scaling exponent at the spin quantum Hall transition and also finds $\eta = \frac{1}{4}$.

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